Int J Theor Phys (2009) 48: 1357–1368
DOI 10.1007/s10773-008-9907-7

Lanczos Potential for the van Stockung Space-Time

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Received: 19 September 2008 / Accepted: 18 November 2008 / Published online: 27 November 2008

Abstract The Lanczos Potential is a theoretical useful tool to find the conformal Weyl curvature tensor C_{abcd} of a given relativistic metric. In this paper we find the Lanczos potential L_{abc} for the van Stockung vacuum gravitational field. Also, we show how the wave equation can be combined with spinor methods in order to find this important three covariant index tensor.

Keywords Lanczos potential theory · Weyl-Lanczos equations · Lanczos coefficients

1 Introduction

The first ideas about Lanczos potential came from 1949 [1]. Nevertheless, is until 1962 when Lanczos suggested that the self-dual part of the Riemann tensor behaves as an auxiliary potential [2]. Through the covariant derivative of Lanczos potential we can obtain the conformal contribution C_{abcd} of the metric curvature. The Weyl tensor can be expressed as first order equations in terms of the Lanczos tensor L_{abc} [3–6]. The task of generating the spacetime Weyl tensor from a tensor potential is known as the Weyl-Lanczos problem and the analogous process for the Riemann curvature tensor is called the Riemann-Lanczos problem.

On the other hand, we have that the field equations of general relativity usually are written as a tensorial equation, where the left side is the Einstein tensor

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}, \tag{1}$$

plus a multiple of the metric λg_{ab}, and the right side is a multiple of the stress tensor $\kappa T_{\mu\nu}$ (where $\kappa = -8\pi G/c^4$). Then, the Einstein's field equations are given as

$$G_{ab} + \lambda g_{ab} = \kappa T_{ab}. \tag{2}$$

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However, there is an alternative formulation of the same set of equations due to Jordan, where this set is expressed as a first order tensorial differential equation in the conformal Weyl tensor [7]. As we mentioned above, the Weyl tensor can be expressed like an expression in first derivative of the Lanczos tensor, then the field equations of Einstein are a second order differential equation in the Lanczos potential formalism. The field equations of Einstein's general relativity are rewritten in Jordan form as follows

$$\begin{aligned} C_{abc}{}^{d}{}_{;d} &= J_{abc}, \\ J_{abc} &= R_{ca;b} - R_{cb;a} + \frac{1}{6} R_{;a} g_{cb} - \frac{1}{6} R_{;b} g_{ca}. \end{aligned} \tag{3}$$

These set, are analogous to Maxwell's equations from electrodynamics

$$F_{a}{}^{b}{}_{;b} = J_{a}. \tag{4}$$

In general relativity the Weyl curvature tensor is very important into the development of the theory, but this is not true with the Lanczos potential, in part because there is a conflict about the non existence of the Lanczos potential in every Riemannian geometry [8]. This is associated with the particular demonstration used by Lanczos in the derivation of the L_{abc} potential [9]. This derivation use the variational principle to arrive at the Bianchi's identities and in the process, it employ some Lagrange multipliers. The potential L_{abc} is identified with these multipliers and there is a general doubt about the generality of this result. In the decade of 1980s Bampi and Caviglia [10] and Illge [11] shown a completely new proof of the existence of L_{abc} in a Riemannian geometry, but this method do not show the way to calculate the L_{abc} [12]. In this work we study the Lanczos Potential from the spinorial perspective for a non rigidly rotating dust solution with cylindrical symmetry, specifically the van Stockung metric [13–16].

The paper is organized as follows, in Sect. 2 we show some algebraic properties of Lanczos potential, in Sect. 3 we study the method for the vacuum space-times. Section 4 is devoted for the spinor formalism for the Lanczos-Weyl equation. In Sect. 5 the spin coefficients for van Stockung space-time are calculated. Later, in Sect. 6 we applied the method of Novello and Velloso for the van Stockung dust solution. Finally, in Sect. 7 we show our conclusions.

2 Algebraic Properties of Lanczos Potential

From the Lanczos Potential a Weyl Candidate Tensor W_{abcd} could be derived from the identity

$$W_{abcd} = L_{ab[c;d]} + L_{cd[a;b]} - {}^{*}L^{*}_{ab[c;d]} - {}^{*}L^{*}_{cd[a;b]}, \tag{5}$$

where * denotes the usual Hodge dual. Now, taking into account the development of the dual part of the covariant derivative of Lanczos potential this could be written as

$$\begin{aligned} W_{abcd} = {} & L_{abc;d} - L_{abd;c} + L_{cdb;a} - L_{cda;b} \\ & + L_{(ad)} g_{bc} + L_{(bc)} g_{ad} - L_{(ac)} g_{bd} \\ & - L_{(bd)} g_{ac} + \frac{2}{3} L^{rs}{}_{r;s} (g_{ac} g_{bd} - g_{ad} g_{bc}), \end{aligned} \tag{6}$$

where we have defined

$$L_{ad} \equiv L_{a\ \ d;r}^{\ r} - L_{a\ \ r;d}^{\ r}. \tag{7}$$

Initially, an arbitrary tensor of third order have 4^3 free components, but further we have to impose the following 40 algebraic symmetries

$$L_{abc} = -L_{bac}, \tag{8}$$

the four conditions (*gauge algebraic conditions of Lanczos*)

$$L_{a\ r}^{\ r} = 0, \tag{9}$$

and the dual four conditions

$${}^*L_{a\ r}^{\ r} = 0 \quad (\text{or} \quad L_{abc} + L_{bca} + L_{cab} = 0), \tag{10}$$

then, the initially Lanczos potential's sixty four degrees of freedom are reduced to sixteen. Further six *differential Lanczos gauge conditions*

$$L_{ab\ \ ;r}^{\ \ r} = 0. \tag{11}$$

In fact, the Lanczos potential also solves a wave equation [13], the tensorial form of this wave equation have proved to be very useful to find the Lanczos potential by an interesting method due to Novello and Velloso [8, 17]. The complete form of the tensorial wave equation for Lanczos potential is

$$\begin{aligned} &\Box L_{abc} + 2R_c^{\ r} L_{bcr} - R_a^{\ r} L_{bcr} - R_b^{\ r} L_{car} \\ &\quad - g_{ac} R^{rs} L_{rbs} + g_{bc} R^{rs} L_{ras} - \frac{1}{2} R L_{abc} = J_{abc}. \end{aligned} \tag{12}$$

In a vacuum space-time, the wave equation for the Lanczos potential (12), is easily written as

$$\Box L_{abc} = 0, \tag{13}$$

this form could be used to outline a possible solution of a non-vacuum space-time. For example, the van Stockung metric is a good candidate for outline a possible Lanczos potential. Further, we can fix integration constants through a set of differential equations that have to satisfy the spinorial version of the Lanczos tensor, given by the relation:

$$L_{ab}^{\ \ c} = L_{A\dot{A}B}^{\ \ \ \ C} \varepsilon_{\dot{B}}^{\ \dot{C}} + \bar{L}_{A\dot{A}\dot{B}}^{\ \ \ \ \dot{C}} \varepsilon_B^{\ C},$$

where ε_{AB} is the usual related *spin-metric* (in the Newman-Penrose convention) [18].

3 Method for the Vacuum Space-Times

If we have a space-time with a global Killing vector field ξ^a [13], is sometimes possible to generate a Lanczos Potential with the following method:

If ξ^a is a non-null Killing vector that satisfies Killing equation

$$\xi_{a;b} + \xi_{b;a} = 0, \tag{14}$$

and also is a hypersurface orthogonal vector

$$\xi_{[a;b}\xi_{c]} = 0,$$

then, is possible to take an unit vector u_a of the group of motion, such that

$$u_a = \frac{\xi_a}{\xi}, \qquad \xi^2 = \epsilon \xi_a \xi^a > 0, \quad \epsilon = \begin{cases} 1, \\ -1. \end{cases}$$

Thus, the Killing equation (14) guaranties that u_a is *expansion-less* and *shear-free*, i.e.

$$u_{(a;b)}(\delta_m{}^a - u^a u_m)(\delta_n{}^b - u^b u_n) = 0,$$

because of the hypersurface orthogonality condition of ξ_a we have

$$u_{[a;b}u_{c]} = 0, \tag{15}$$

and then

$$u_{a;b} = \epsilon a_a u_b, \tag{16}$$

where we have defined the first curvature vector of the group of congruence (also called and known as acceleration) to be $a_a = u_{a;b}u^b$, which for all group of motions even those not satisfying (15), is a gradient $a_a = (\ln(1/\xi))_{,a}$. Then, a candidate of Lanczos potential is given by

$$L^{(1)}_{abc} = (a_a u_b - a_b u_a)u_c - \frac{1}{3}\epsilon(a_a g_{bc} - a_b g_{ac}), \tag{17}$$

which satisfies (8), (9) and (10). In this way, verifying the Lanczos gauge (11) and the condition of the Potential (6), we can fix completely our candidate as a full-fledged Lanczos Potential. In the next section we will trace the spinorial analog of (6) to fix our candidate.

4 The Spinor Formalism for the Lanczos-Weyl Equation

In a spinor dyad

$$\varepsilon_{(0)}{}^A = o^A, \qquad \varepsilon_{(1)}{}^A = \iota^A, \qquad \varepsilon_A{}^{(0)} = -\iota_A, \qquad \varepsilon_A{}^{(1)} = o_A,$$

we can write the Weyl tensor in spinorial form as

$$C_{abcd} = \Psi_{(ABCD)}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{C}\dot{D}} + \overline{\Psi}_{(\dot{A}\dot{B}\dot{C}\dot{D})}\varepsilon_{AB}\varepsilon_{CD}.$$

Then, the spinorial form of the Weyl-Lanczos relation (6), is written as follows

$$\Psi_{ABCD} = 2L_{ABC\dot{R};D}{}^{\dot{R}}. \tag{18}$$

With the aim of write this expression in terms of the spinorial coefficients, we should project it, at some null basis. Thus we have

$$\begin{aligned} \Psi_{(a)(b)(c)(d)} &= \Psi_{ABCD}\varepsilon_{(a)}{}^A\varepsilon_{(b)}{}^B\varepsilon_{(c)}{}^C, \varepsilon_{(d)}{}^D, \\ &= 2L_{ABC\dot{R};D}{}^{\dot{R}}\varepsilon_{(a)}{}^A\varepsilon_{(b)}{}^B\varepsilon_{(c)}{}^C\varepsilon_{(d)}{}^D, \end{aligned} \tag{19}$$

and developing terms, we find that

$$\begin{aligned}\frac{1}{2}\Psi_{(a)(b)(c)(d)} = {} & L_{(b)(c)(d)(\dot{r});(a)(\dot{s})}\varepsilon^{(\dot{r})(\dot{s})} \\ & - \varepsilon^{(\dot{r})(\dot{s})}L_{(t)(c)(d)(\dot{s})}\gamma_{(a)(\dot{r})(b)}{}^{(t)} \\ & - \varepsilon^{(\dot{r})(\dot{s})}L_{(b)(t)(d)(\dot{s})}\gamma_{(a)(\dot{r})(c)}{}^{(t)} \\ & - \varepsilon^{(\dot{r})(\dot{s})}L_{(b)(c)(t)(\dot{s})}\gamma_{(a)(\dot{r})(d)}{}^{(t)} \\ & + \varepsilon^{(\dot{r})(\dot{s})}L_{(b)(c)(d)(\dot{s})}\overline{\gamma}_{(a)(\dot{t})(\dot{r})}{}^{(\dot{t})},\end{aligned} \tag{20}$$

where the spin-coefficients are defined as [18]:

$$\gamma_{(a)(\dot{a})(b)}{}^{(c)} = \varepsilon_{A}{}^{(c)}\varepsilon_{(b)}{}^{A}{}_{;(a)(\dot{a})} = -\varepsilon_{(b)}{}^{A}\varepsilon_{A}{}^{(c)}{}_{;(a)(\dot{a})}.$$

The Weyl scalar invariants are

$$\begin{aligned}&\Psi_0 = \Psi_{(0)(0)(0)(0)}, \qquad \Psi_1 = \Psi_{(0)(0)(0)(1)},\\ &\Psi_2 = \Psi_{(0)(0)(1)(1)}, \qquad \Psi_3 = \Psi_{(0)(1)(1)(1)},\\ &\Psi_4 = \Psi_{(1)(1)(1)(1)}.\end{aligned} \tag{21}$$

And the corresponding Lanczos scalars

$$\begin{aligned}&L_0 = L_{(0)(0)(0)(\dot{0})}, \qquad L_1 = L_{(0)(0)(1)(\dot{0})},\\ &L_2 = L_{(0)(1)(1)(\dot{0})}, \qquad L_3 = L_{(1)(1)(1)(\dot{0})},\\ &L_4 = L_{(0)(0)(0)(\dot{1})}, \qquad L_5 = L_{(0)(0)(1)(\dot{1})},\\ &L_6 = L_{(0)(1)(1)(\dot{1})}, \qquad L_7 = L_{(1)(1)(1)(\dot{1})}.\end{aligned} \tag{22}$$

If we write (20) in the Newman-Penrose convention of spinorial coefficients, we have that [19]

$$\begin{aligned}&\frac{1}{2}\Psi_0 = (-\delta+\bar{\alpha}+3\beta-\bar{\pi})L_0 + (D-3\varepsilon+\bar{\varepsilon}-\bar{\rho})L_4 - 3\sigma L_1 + 3\kappa L_5,\\ &\frac{1}{2}\Psi_1 = (-\delta+\bar{\alpha}+\beta-\bar{\pi})L_1 + (D+\bar{\varepsilon}-\varepsilon-\bar{\rho})L_5 + \mu L_0 - 2\sigma L_2 - \pi L_4 + 2\kappa L_6,\\ &\frac{1}{2}\Psi_1 = (-D'-\bar{\mu}+3\gamma+\bar{\gamma})L_0 + (\delta'-3\alpha+\bar{\beta}-\bar{\tau})L_4 - 3\tau L_1 + 3\rho L_5,\\ &\frac{1}{2}\Psi_2 = (-\delta+\bar{\alpha}-\beta-\bar{\pi})L_2 + (D+\bar{\varepsilon}+\varepsilon-\bar{\rho})L_6 + 2\mu L_1 - \sigma L_3 - 2\pi L_5 + \kappa L_7,\\ &\frac{1}{2}\Psi_2 = (-D'+\gamma+\bar{\gamma}-\bar{\mu})L_1 + (\delta'-\alpha+\bar{\beta}-\bar{\tau})L_5 + \nu L_0 - 2\tau L_2 - \lambda L_4 + 2\rho L_6,\\ &\frac{1}{2}\Psi_3 = (-D'-\bar{\mu}+\bar{\gamma}-\gamma)L_2 + (\delta'+\alpha+\bar{\beta}-\bar{\tau})L_6 + 2\nu L_1 - \tau L_3 - 2\lambda L_5 + \rho L_7,\\ &\frac{1}{2}\Psi_3 = (-\delta+\bar{\alpha}-3\beta-\bar{\pi})L_3 + (D+3\varepsilon+\bar{\varepsilon}-\bar{\rho})L_7 + 3\mu L_2 - 3\pi L_6,\\ &\frac{1}{2}\Psi_4 = (-D'-\bar{\mu}-3\gamma+\bar{\gamma})L_3 + (\delta'+3\alpha+\bar{\beta}-\bar{\tau})L_7 + 3\nu L_2 - 3\lambda L_6.\end{aligned} \tag{23}$$

Now, we can introduce the null tetrad induced by the spin dyad as

$$\begin{aligned} l^a &= o^A o^{\dot{A}}, \qquad m^a = o^A \iota^{\dot{A}}, \\ \bar{m}^a &= \iota^A o^{\dot{A}}, \qquad n^a = \iota^A \iota^{\dot{A}}, \end{aligned} \tag{24}$$

then the differential operators in (23) for an arbitrary scalar f are

$$\begin{aligned} Df &\equiv f_{;0\dot{0}} = o^A o^{\dot{A}} f_{;A\dot{A}} = l^a f_{;a} = \bar{D} f, \\ \delta f &\equiv f_{;0\dot{1}} = o^A \iota^{\dot{A}} f_{;A\dot{A}} = m^a f_{;a} = \bar{\delta}' f, \\ \delta' f &\equiv f_{;1\dot{0}} = \iota^A o^{\dot{A}} f_{;A\dot{A}} = \bar{m}^a f_{;a} = \bar{\delta} f, \\ D' f &\equiv f_{;1\dot{1}} = \iota^A \iota^{\dot{A}} f_{;A\dot{A}} = n^a f_{;a} = \bar{D}' f, \end{aligned} \tag{25}$$

with the *primed* (′) Penrose operator is defined as [18]:

$$\begin{aligned} (o^A)' &= i\iota^A, \qquad (\iota^A)' = i o^A, \\ (o^{\dot{A}})' &= -i\iota^{\dot{A}}, \qquad (\iota^{\dot{A}})' = -i o^{\dot{A}}, \end{aligned}$$

and

$$\begin{aligned} (l^a)' &= n^a, \qquad (n^a)' = l^a, \\ (m^a)' &= \bar{m}^a, \qquad (\bar{m}^a)' = m^a. \end{aligned}$$

In terms of this null tetrad, the Weyl and Lanczos scalars can be rewritten as [19]

$$\begin{aligned} &\Psi_0 = C_{abcd} l^a m^b l^c m^d, \qquad \Psi_1 = C_{abcd} l^a m^b l^c n^d, \\ &\Psi_2 = C_{abcd} l^a m^b \bar{m}^c n^d, \qquad \Psi_3 = C_{abcd} l^a n^b \bar{m}^c n^d, \\ &\Psi_4 = C_{abcd} \bar{m}^a n^b \bar{m}^c n^d. \end{aligned} \tag{26}$$

and

$$\begin{aligned} L_0 &= L_{abc} l^a m^b l^c, \qquad L_1 = L_{abc} l^a m^b \bar{m}^c, \\ L_2 &= L_{abc} \bar{m}^a n^b l^c, \qquad L_3 = L_{abc} \bar{m}^a n^b \bar{m}^c, \\ L_4 &= L_{abc} l^a m^b m^c, \qquad L_5 = L_{abc} l^a m^b n^c, \\ L_6 &= L_{abc} \bar{m}^a n^b m^c, \qquad L_7 = L_{abc} \bar{m}^a n^b n^c. \end{aligned} \tag{27}$$

5 The Spin Coefficients for the van Stockung Space-Time

In this section we consider the van Stockung space-time which is a non rigidly rotating dust solution with cylindrical symmetry given by [16]

$$ds^2 = \mathrm{e}^{-a^2 r^2}(dr^2 + dz^2) + r^2 d\theta^2 - (dt + ar^2 d\theta)^2. \tag{28}$$

This solution is clearly for non-vacuum $R_{ab} \neq 0$. Nevertheless, we can outline an approach to find a solution for the Lanczos Potential from a very slight modified method due to Novello and Velloso. As we will prove, we only have to find some constants in the functional form of the Lanczos Potential obtained from this method in the spinorial formulation. For the van Stockung space-time we can construct a Newman-Penrose null tetrad as follows

$$\begin{aligned}
\mathbf{e}_{(0)}{}^{a} &= l^a = (0, 0, -e^{a^2r^2/2}, 1),\\
\mathbf{e}_{(1)}{}^{a} &= n^a = \frac{1}{2}(0, 0, e^{a^2r^2/2}, 1),\\
\mathbf{e}_{(2)}{}^{a} &= m^a = \frac{1}{\pm\sqrt{2}}\Big(ie^{a^2r^2/2}, \frac{1}{r}, 0, -ar\Big),\\
\mathbf{e}_{(3)}{}^{a} &= \bar{m}^a = \frac{1}{\pm\sqrt{2}}\Big(-ie^{a^2r^2/2}, \frac{1}{r}, 0, -ar\Big),
\end{aligned}\tag{29}$$

where the double sign of the radical $\pm\sqrt{2}$ is an indetermination to be solved. Then, we have further determine this sign when we compute the Weyl scalars (36). The basis $\{\mathbf{e}_{(a)}{}^{b}\}$ clearly satisfies $\eta^{(a)(b)}\mathbf{e}_{(a)}{}^{c}\mathbf{e}_{(b)}{}^{d} = g^{cd}$, and the spinorial coefficients for this metric could be written as follows

$$\begin{aligned}
\alpha &= \frac{1}{2\sqrt{2}r} i \exp\Big(\frac{a^2r^2}{2}\Big),\\
\beta &= \frac{1}{2\sqrt{2}r} i \exp\Big(\frac{a^2r^2}{2}\Big),\\
\gamma &= -\frac{1}{4} ia \exp\Big(\frac{a^2r^2}{2}\Big),\\
\varepsilon &= -\frac{1}{2} ia \exp\Big(\frac{a^2r^2}{2}\Big),\\
\kappa &= -\frac{1}{\sqrt{2}} ia^2r \exp\Big(\frac{a^2r^2}{2}\Big),\\
\lambda &= 0,\\
\mu &= -\frac{1}{2} ia \exp\Big(\frac{a^2r^2}{2}\Big),\\
\nu &= -\frac{1}{4\sqrt{2}} ia^2r \exp\Big(\frac{a^2r^2}{2}\Big),\\
\pi &= \frac{1}{2\sqrt{2}} ia^2r \exp\Big(\frac{a^2r^2}{2}\Big),\\
\rho &= -ia \exp\Big(\frac{a^2r^2}{2}\Big),\\
\sigma &= 0,\\
\tau &= \frac{1}{2\sqrt{2}} ia^2r \exp\Big(\frac{a^2r^2}{2}\Big).
\end{aligned}\tag{30}$$

These spinorial coefficients are useful to calculate the Lanczos spinorial scalars of expression (23). By computing also the Weyl scalars in the same relation, we can obtain a set of 8 differential equations for the 8 Lanczos spinorial scalar functions $L_0, L_1, \dots, L_7$.

But finding a solution of the system (23) could be daunting in principle. Then, in Sect. 6, we will use another more easy approach in order to find the spinorial scalars components of

Lanczos potential. In this way we can test an ansatz for the unknown functions $L_0, \dots, L_7$, later we substitute these in the system (23) and finally, we only fix some simple constants to get it satisfied.

6 Application of the Novello and Velloso Method for the van Stockung Solution

The axial symmetric Killing vector for the van Stockung space-time (28) can be chosen as $\xi^a = (0,0,0,1)$, (taking into account the order of the coordinate system as follows: $x^a = (t, r, \theta, z)$), then we have $\xi_a = (0,0,0,e^{-a^2r^2})$, and $\xi^2 = \xi_a\xi^a = e^{-a^2r^2} \Rightarrow \epsilon = 1$. Also the unit tangent vector for the group of motions (or the *velocity* vector) is

$$u_a = \frac{1}{\xi}\xi_a = (0,0,0,e^{-a^2r^2/2}),$$

and the acceleration vector is

$$a_a = u_{a;r}u^r = (0, a^2r, 0, 0),$$

then Lanczos Potential's candidate for the van Stockung space-time is given by its relevant components (see (17) and remember the algebraic relation of skew-symmetry in the first two indices)

$$\begin{aligned}
L^{(1)}_{r\theta\theta} &= -L^{(1)}_{\theta r\theta} = \frac{1}{3}a^2r^3(a^2r^2 - 1),\\
L^{(1)}_{rt\theta} &= -L^{(1)}_{tr\theta} = \frac{1}{3}a^3r^3,\\
L^{(1)}_{rzz} &= -L^{(1)}_{zrz} = \frac{2}{3}a^2re^{-a^2r^2},\\
L^{(1)}_{r\theta t} &= -L^{(1)}_{\theta rt} = \frac{1}{3}a^3r^3,\\
L^{(1)}_{rtt} &= -L^{(1)}_{trt} = \frac{1}{3}a^2r.
\end{aligned} \tag{31}$$

While the respective components of conformal Weyl tensor for the metric (28) are

$$\begin{aligned}
C_{r\theta r\theta} &= \frac{1}{3}a^2r^2(2 + 7a^2r^2),\\
C_{r\theta rt} &= \frac{4}{3}a^3r^2,\\
C_{rzrz} &= -\frac{1}{3}a^2e^{-a^2r^2},\\
C_{rtrt} &= \frac{1}{3}a^2,\\
C_{\theta z\theta z} &= -\frac{1}{3}a^2r^2(1 + 8a^2r^2),\\
C_{\theta zzt} &= \frac{5}{3}a^3r^2,\\
C_{\theta t\theta t} &= \frac{1}{3}a^2r^2e^{a^2r^2},\\
C_{ztzt} &= -\frac{2}{3}a^2.
\end{aligned} \tag{32}$$

From these relations we can compute the inner product indicated by (26) using (29) in order to find the Weyl scalars candidates for the van Stockung space-time. These, could be written as

$$\begin{aligned}
&\Psi_0 = 0, \qquad \Psi_1 = -\frac{1}{2}c_1\sqrt{2}a^3 r e^{a^2r^2}, \\
&\Psi_2 = -\frac{1}{3}c_2 a^2 e^{a^2r^2}, \qquad \Psi_3 = -\frac{1}{4}c_3\sqrt{2}a^3 r e^{a^2r^2}, \\
&\Psi_4 = 0, \qquad c_1, c_2, c_3 = \begin{cases} 1, \\ -1, \end{cases}
\end{aligned} \tag{33}$$

where we have to fix the constants c_1, c_2, c_3 as 1 or -1, due to the indetermination of the radical sign that appears in the expressions.

Similarly from (31), using (29) and (27) we can get the following Lanczos spinorial scalars candidates

$$\begin{aligned}
&L_0 = -\frac{1}{2}k_0\sqrt{2}ia^2 r e^{a^2r^2/2}, \qquad L_1 = 0, \\
&L_2 = \frac{1}{12}k_2\sqrt{2}ia^2 r e^{a^2r^2/2}, \qquad L_3 = 0, \\
&L_5 = \frac{1}{12}k_5\sqrt{2}ia^2 r e^{a^2r^2/2}, \qquad L_4 = 0, \\
&L_7 = -\frac{1}{8}k_7\sqrt{2}ia^2 r e^{a^2r^2/2}, \qquad L_6 = 0,
\end{aligned} \tag{34}$$

where again, we have to fix the constants k_0, k_2, k_5, k_7 in order to determine which are the final Lanczos spinorial scalars. This final task can be accomplished with the help of the Weyl-Lanczos differential equations (23).

The differential operators in (23) for an arbitrary scalar function f of the coordinates independent variables r, θ, z, t are

$$\begin{aligned}
Df &= -e^{a^2r^2/2}\frac{\partial f}{\partial z} + \frac{\partial f}{\partial t}, \\
D'f &= \frac{1}{2}e^{a^2r^2/2}\frac{\partial f}{\partial z} + \frac{1}{2}\frac{\partial f}{\partial t}, \\
\delta f &= \frac{1}{\sqrt{2}}ie^{a^2r^2/2}\frac{\partial f}{\partial r} + \frac{1}{\sqrt{2}r}\frac{\partial f}{\partial \theta} - \frac{1}{\sqrt{2}}ar\frac{\partial f}{\partial t}, \\
\delta' f &= -\frac{1}{\sqrt{2}}ie^{a^2r^2/2}\frac{\partial f}{\partial r} + \frac{1}{\sqrt{2}r}\frac{\partial f}{\partial \theta} - \frac{1}{\sqrt{2}}ar\frac{\partial f}{\partial t}.
\end{aligned} \tag{35}$$

After the substitution of (35) in (23), we have the following system

$$\begin{aligned}
0 &= \frac{1}{4}a^4r^2e^{a^2r^2}(-k_0+k_5),\\
-\frac{1}{4}\sqrt{2}c_1a^3re^{a^2r^2} &= -\frac{1}{4}\sqrt{2}k_0a^3re^{a^2r^2},\\
-\frac{1}{4}\sqrt{2}c_1a^3re^{a^2r^2} &= \frac{1}{4}\sqrt{2}a^3re^{a^2r^2}(-2k_0+k_5),\\
-\frac{1}{6}c_2a^2e^{a^2r^2} &= \frac{1}{24}a^2e^{a^2r^2}(4k_2+k_2a^2r^2\\
&\quad +2k_5a^2r^2-3k_7a^2r^2),\\
-\frac{1}{6}c_2a^2e^{a^2r^2} &= \frac{1}{24}a^2e^{a^2r^2}(4k_5+k_5a^2r^2\\
&\quad +2k_2a^2r^2-3k_0a^2r^2),\\
-\frac{1}{8}\sqrt{2}c_3a^3re^{a^2r^2} &= -\frac{1}{8}\sqrt{2}k_7a^3re^{a^2r^2},\\
-\frac{1}{8}\sqrt{2}c_3a^3re^{a^2r^2} &= \frac{1}{8}\sqrt{2}a^3re^{a^2r^2}(-2k_7+k_2),\\
0 &= \frac{1}{16}a^4r^2e^{a^2r^2}(-k_7+k_2).
\end{aligned} \tag{36}$$

A solution to this system of equations is given by

$$\begin{aligned}
&c_1=k_0, \qquad c_2=-k_0, \qquad c_3=k_0,\\
&k_2=k_0, \qquad k_5=k_0, \qquad k_7=k_0,
\end{aligned} \tag{37}$$

Now, if we fix the only one remaining constant k_0 to be -1, then we can fix all the constants in (33) and (34). By doing this, we can see that the radical expressions in (33) and (34), have been chosen with the negative root of the radical $\sqrt{2}$, in order to be consistent with the differential spinorial expressions (23).

The Lanczos potential have spinorial scalars components given by

$$\begin{aligned}
L_0 &= \frac{1}{2}\sqrt{2}ia^2re^{a^2r^2/2}, & L_1 &= 0,\\
L_2 &= -\frac{1}{12}\sqrt{2}ia^2re^{a^2r^2/2}, & L_3 &= 0,\\
L_5 &= -\frac{1}{12}\sqrt{2}ia^2re^{a^2r^2/2}, & L_4 &= 0,\\
L_7 &= \frac{1}{8}\sqrt{2}ia^2re^{a^2r^2/2}, & L_6 &= 0,
\end{aligned} \tag{38}$$

with a set of Weyl scalars chosen as

$$\begin{aligned}
\Psi_0 &= 0, & \Psi_1 &= \frac{1}{2}\sqrt{2}a^3re^{a^2r^2},\\
\Psi_2 &= -\frac{1}{3}a^2e^{a^2r^2}, & \Psi_3 &= \frac{1}{4}\sqrt{2}a^3re^{a^2r^2},\\
\Psi_4 &= 0.
\end{aligned} \tag{39}$$

The solution can be written in a more compact form: $L_r = 0$ with $r = 1, 3, 4, 6$ and $L_0 = -\kappa$, $L_2 = -\tau/3$, $L_5 = -\tau/3$, $L_7 = -\upsilon$, that is, L_r can be written in terms of spin coefficients [20, 21], given in (30). Thus, our task of finding the Lanczos potential for the dust solution of van Stockung (28) has been done.

7 Concluding Remarks

We have found that the method of Novello and Velloso [8] described in Sect. 3, could be used in conjunction with the set of equations (23) found by Ares de Parga et al. [19], and that are the spinorial version of the primordial relation that keeps Lanczos potential with the Weyl conformal curvature tensor given in (5). We have also seen, that fixing some constants in an ansatzs found with the first method, the work of finding a solution to (23) is simplified enormously. Then, we could appreciate the theoretical importance of Lanczos potential in deriving the Weyl curvature tensor. As its electromagnetic counterpart A_a, by fixing until some tensor constant χ_{abc}, the Lanczos potential can be written in general like $L_{abc} = xL^{(1)}_{abc} + \chi_{abc}$. Therefore in theory, it is important to choose some gauge conditions that fix this potential as can be stated by the algebraic (9) and differential (11) Lanczos gauge conditions.

Acknowledgements This work was done by financial support of Research Projects SIP-20082788 and CONACYT-91335, also COFAA and EDI-IPN grants. The authors thank to Prof. Alfredo López for his suggestions and comments to improve the manuscript.

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